Problem S1 Solution

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = \frac{dx}{dt} + cx$$

i) $x = e^{st}$ $\qquad y = H(s)e^{st}$

$$\frac{d^2}{dt^2}\left(H(s)e^{st}\right) + a\frac{d}{dt}\left(H(s)e^{st}\right) + bH(s)e^{st} = \frac{d}{dt}e^{st} + ce^{st}$$

ii)

$$H(s)s^2\cancel{e^{st}} + aH(s)s\cancel{e^{st}} + bH(s)\cancel{e^{st}} = s\cancel{e^{st}} + c\cancel{e^{st}}$$

$$H(s)\left[s^2 + as + b\right] = s + c$$

$$H(s) = \frac{s+c}{s^2+as+b}$$

iii)

$$H(j\omega) = \frac{j\omega + c}{-\omega^2 + aj\omega + b} = \frac{j\omega + c}{aj\omega + (b - \omega^2)}$$

Magnitude is

$$|H(j\omega)| = \sqrt{H(j\omega)\cdot H^*(j\omega)}$$

$$|H(j\omega)| = \sqrt{\frac{(j\omega + c)}{(aj\omega + (b-\omega^2))} \cdot \frac{(-j\omega + c)}{(-aj\omega + (b-\omega^2))}}$$

$$= \sqrt{\frac{\omega^2 + c^2}{(a\omega)^2 + (b-\omega^2)^2}}$$

To find the angle we need the real and imaginary parts of $H(j\omega)$. Multiply numerator and denominator by complex conjugate of denominator

$$\frac{\omega^3}{(a-c)\omega^2} = \frac{-\omega}{(a-c)}$$

$$H(j\omega) = \frac{(j\omega + c)}{(aj\omega + (b-\omega^2))} \cdot \frac{(-aj\omega + (b-\omega^2))}{(-aj\omega + (b-\omega^2))}$$

$$= \frac{[a\omega^2 + c(b-\omega^2)] + [\omega(b-\omega^2) - ca\omega]j}{(a\omega)^2 + (b-\omega^2)^2}$$

$$\measuredangle H(j\omega) = \tan^{-1}\left[\frac{\mathrm{Im}(H(j\omega))}{\mathrm{Re}(H(j\omega))}\right] = \tan^{-1}\left[\frac{\omega(b-\omega^2) - ca\omega}{a\omega^2 + c(b-\omega^2)}\right]$$